Section 10.5

The Direct Comparison Test

Our first convergence test in this section mimics the comparison test for improper integrals:

Theorem 10.5.1. The Direct Comparison Test
Let $\sum a_k$ and $\sum b_k$ be series with positive terms.

1. If $a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
2. If $b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Example 10.5.2.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{2k^4 - 1}$

2. $\displaystyle\sum_{k=2}^{\infty} \frac{\ln k}{k^3}$

The Limit Comparison Test

The Direct Comparison Test is effective when there is an obvious series for the comparison, but it generally requires some finesse of inequalities. A more powerful test is the limit comparison test:

Theorem 10.5.3. The Limit Comparison Test
Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L.$$

1. If $0 < L < \infty$, then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
2. If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
3. If $L \to \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Example 10.5.4.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+1}$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2-1}{k^3+4}$

5. $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{2}{k}\right)^k$

6. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

7. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k10^k}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{e^k - 1}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{e^{2k} + k}{e^{5k} - k^2}$